

# Null Geodesics in Five-Dimensional Manifolds

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## Abstract

We analyze a class of 5D non-compact warped-product spaces characterized by metrics that depend on the extra coordinate via a conformal factor. Our model is closely related to the so-called canonical coordinate gauge of Mashhoon et al. We confirm that if the 5D manifold in our model is Ricci-flat, then there is an induced cosmological constant in the 4D sub-manifold. We derive the general form of the 5D Killing vectors and relate them to the 4D Killing vectors of the embedded spacetime. We then study the 5D null geodesic paths and show that the 4D part of the motion can be timelike — that is, massless particles in 5D can be massive in 4D. We find that if the null trajectories are affinely parameterized in 5D, then the particle is subject to an anomalous acceleration or fifth force. However, this force may be removed by reparameterization, which brings the correct definition of the proper time into question. Physical properties of the geodesics — such as rest mass variations induced by a variable cosmological “constant”, constants of the motion and 5D time-dilation effects — are discussed and are shown to be open to experimental or observational investigation.

Keywords: general relativity — non-compactified Kaluza-Klein theory — particle dynamics

## I. INTRODUCTION

The extension of 4D spacetime to higher dimensions is now commonplace, as in Kaluza-Klein theory (5D), superstrings (10D) and supergravity (11D). There is currently a large amount of interest in brane-world theories with non-compact extra dimensions serving as a possible route to reconciling the formalisms used to describe particle and gravitational physics [1,2]. Recent papers have presented and analyzed new exact solutions of the 5D vacuum field equations in the context of 4D wormholes [3] and Friedmann-Robertson-Walker cosmologies [4]. The motion of test particles in 5D has been much studied [5–15]. The five dimensional geodesic equation can be reduced to 4D equations of motion and an equation governing the motion in the extra dimension. As in 4D general relativity, the precise form of these formulae depends on whether we are considering timelike  $dS^2 > 0$ , null  $dS^2 = 0$  or

spacelike  $d\mathcal{S}^2 < 0$  trajectories, where  $d\mathcal{S}^2 = g_{AB}dx^A dx^B$  is the 5D arclength<sup>1</sup>. The possibility that the dynamics of particles in Kaluza-Klein theories could involve spacelike 5D paths was raised by Davidson and Owen [8], the argument being that the 4D part of the trajectory could be a casual curve  $g_{\alpha\beta}dx^\alpha dx^\beta > 0$  even if the higher dimensional trajectory is acausal. A common feature of the derived 4D equations of motion is that they do not appear to be spacetime geodesics [14]. That is, there is in general an anomalous acceleration in 4D due to the fifth dimension, or equivalently a fifth force. This has so far not been observed, either in local dynamics or cosmology [16]. Recently, Liu and Mashoon have interpreted this extra force as being related to variations in the rest masses of test particles traveling on 5D timelike, null and spacelike geodesics [15].

In this paper, we examine the nature of this anomalous acceleration in detail using a 5D model that is conformally related to the manifold first introduced by Kaluza [17]. In section II, we demonstrate that if the 5D vacuum field equations are enforced, the 4D part of the metric satisfies Einstein's equations in the absence of ordinary matter with a cosmological constant  $\Lambda > 0$ . In section III, we derive the general form of Killing vectors of the 5D manifold  $\xi^A$  and relate them to Killing vectors of the 4D  $y = \text{constant}$  sub-manifold  $\eta^\alpha$ . Higher dimensional particle dynamics is studied in section IV using the assumption that all trajectories are 5D null geodesics, a choice motivated by the special relativistic relation  $E^2 = p^2 + m^2$ . It is then shown that particles can travel along timelike 4D paths even if  $d\mathcal{S}^2 = 0$ . We find that if the parameterization of the 5D null curves is affine, the 4D part of the trajectory is subject to an acceleration parallel to the 4-velocity. However, this anomalous force can be removed by a parameter transformation, which introduces ambiguities in defining the 4D proper time. This has important consequences for the determination of variations in rest mass, which are discussed in section V A. In the 5D Ricci-flat case, we show how rest-mass variations can arise from an exchange of energy between the particle and the vacuum. The topic of section V B is quantities conserved along the geodesics, while section V C discusses time-dilation effects associated with different parameterizations and potential experiments to determine the “true” proper time.

## II. THE 5D METRIC AND THE UNIQUENESS OF CANONICAL COORDINATES

We will study the geodesic motion of particles in a 5D manifold using a particularly useful coordinate gauge. Our choice of coordinates is based on the 5D canonical metric introduced by Mashhoon et al. [10,12–14]. The line element in canonical coordinates is given by

$$d\mathcal{S}^2 = \frac{y^2}{L^2} g_{\alpha\beta}(x^\mu, y) dx^\alpha dx^\beta - dy^2. \quad (2.1)$$

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<sup>1</sup>In this paper, we label 5D coordinates  $x^A$  by uppercase Latin indices that run 0 — 4 with  $x^4 = y$ . Lowercase Greek letters run over spacetime indices 0 — 3. We employ units where  $G = c = 1$ . The signature of the 5D metric is  $(+ - - - -)$  while the signature of the 4D metric is  $(+ - - -)$ .

Here,  $L$  is a constant introduced to give  $d\mathcal{S}^2$  the correct units. The metric (2.1) is general in the sense that the line element in any 5D manifold may be expressed in the canonical form via appropriate coordinate transformations. This choice of gauge results in great algebraic simplification of the vacuum 5D field equations, which identify the constant  $L$  with an induced 4D cosmological constant via  $\Lambda = 3/L^2$ .

The manifold that we examine in this paper is represented by the 5D line element

$$d\mathcal{S}^2 = \Phi^2(y)g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta - dy^2. \quad (2.2)$$

This is an example of a so-called “warped-product space” which has received a fair amount of recent attention in the literature [1,2]. Here,  $\Phi(y)$  is an unspecified function of  $y$ , which we call the *conformal prefactor*. We will use the notation

$$\Omega(y) \equiv \Phi^2(y) \quad (2.3)$$

where convenient (both notations are common in the literature). Our 5D model is obviously similar to (2.1), but there are two notable exceptions: we do not restrict  $\Phi(y) = y/L$ , and  $g_{\alpha\beta}$  is assumed to depend on spacetime variables  $x^\mu$  only. This metric (2.2) is not general and in fact refers to a set of 5D manifolds with a certain type of symmetry. We can elucidate this symmetry by performing a conformal transformation

$$g_{AB} \rightarrow \Phi^{-2}(y)g_{AB}, \quad (2.4)$$

followed by the coordinate transformation

$$Y = \int^y \Phi^{-1}(u) du. \quad (2.5)$$

The line element  $d\hat{\mathcal{S}}$  in the conformal manifold is then given by

$$d\hat{\mathcal{S}}^2 = g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta - dY^2. \quad (2.6)$$

This is the classic form of the metric of a 5D manifold in the absence of electromagnetic potentials  $A^\alpha$  [17]. Thus the 5D  $y$ -dependent spaces (2.2) are related to ordinary 4D spaces (2.6) via a simple conformal transformation.

These comments imply that all the information about the conformal 5D manifold is embedded in  $g_{\alpha\beta}$ . It is for this reason that we call  $g_{\alpha\beta}$  the 4D *conformal metric*. Now, the *induced metric* on  $y = \text{constant}$  4D hypersurfaces  $\Sigma_y$  is

$$h_{\alpha\beta} = \Phi^2(y)g_{\alpha\beta}. \quad (2.7)$$

Because the difference between the two 4D metrics is a  $y$ -dependent prefactor, both  $g_{\alpha\beta}$  and  $h_{\alpha\beta}$  transform as 4-tensors on  $\Sigma_y$  and both satisfy completeness relations. We will use  $g_{\alpha\beta}$  to raise and lower indices on 4D objects (for example, the projection of particle velocities onto  $\Sigma_y$ ). Because all the  $y$ -dependence of the induced metric is concentrated in the conformal prefactor, the 4D Christoffel symbols and derived curvature quantities defined for each of the 4D metrics are equivalent and independent of  $y$ . For all intents and purposes,  $g_{\alpha\beta}$  is the fundamental quantity on  $\Sigma_y$ .

In most of this paper we will not assume any particular form of the 5D field equations. However, it is useful to make contact with previous work by assuming, like other authors [10,12–14], that the 5D vacuum field equations are

$$R_{AB} = 0, \quad A, B = 0, 1, 2, 3, 4. \quad (2.8)$$

We will now prove that given a metric of the form (2.2) and the field equations (2.8), then the conformal prefactor  $\Phi(y)$  is determined up to an integration constant and a linear translation on  $y$ .

It is well known that the 15 equations (2.8) can be broken down into a set of 10 Einstein equations, a set of 4 Maxwell equations and a wave equation [14]. For the metric (2.2), there are no electromagnetic potentials ( $g_{4\alpha} = 0$ ) and the scalar field is a constant ( $g_{44} = -1$ ). It is then straightforward to extract the 4D part of (2.8), which yields

$$R_{\alpha\beta} = -\frac{1}{2} \left( \Omega'' + \Omega^{-1} \Omega'^2 \right) g_{\alpha\beta}, \quad (2.9)$$

where a prime denotes differentiation with respect to  $y$ . This is from the  $A, B = 0, 1, 2, 3$  components of (2.9). Here,  $R_{\alpha\beta} = 0$  is the 4D Ricci tensor defined with respect to either the induced metric  $h_{\alpha\beta}(x^\mu, y)$  or the conformal metric  $g_{\alpha\beta}(x^\mu)$ . The  $R_{4\alpha} = 0$  parts of (2.9) are automatically satisfied because  $g_{4\alpha} = 0$ . The 44-component of (2.8) yields

$$\Omega'' - \frac{1}{2} \Omega^{-1} \Omega'^2 = 0. \quad (2.10)$$

If we now contract the 4D relation (2.9), we obtain the 4D Ricci scalar as

$$R = -2 \left( \Omega'' + \Omega^{-1} \Omega'^2 \right). \quad (2.11)$$

However, the left-hand side is a function of spacetime variables  $x^\mu$  while the right-hand side is a function of  $y$  only. Hence, both sides must be equal a constant. We choose

$$R = -4\Lambda, \quad (2.12)$$

$$\Lambda = \frac{1}{2} \left( \Omega'' + \Omega^{-1} \Omega'^2 \right). \quad (2.13)$$

Then (2.9) gives for the 4D Ricci and Einstein tensors

$$R_{\alpha\beta} = -\Lambda g_{\alpha\beta} = -\Lambda \Omega^{-1}(y) h_{\alpha\beta}, \quad (2.14)$$

$$G_{\alpha\beta} = +\Lambda g_{\alpha\beta} = +\Lambda \Omega^{-1}(y) h_{\alpha\beta}. \quad (2.15)$$

For observers restricted to  $\Sigma_y$  hypersurfaces, these are the conventional equations of general relativity in the absence of ordinary matter, but with a finite cosmological constant. [Equivalently, they describe a vacuum state with a pressure and density that obeys  $p = -\rho$  as in the de Sitter model.] We will discuss the experiences of freely-falling observers below in section V A. Eliminating the first-derivative terms in (2.10) and (2.13) yields

$$\Omega'' = \frac{2}{3}\Lambda \quad \Rightarrow \quad \Omega(y) = \frac{1}{3}\Lambda(y - y_*)^2 + k, \quad (2.16)$$

where  $y_*$  and  $k$  are arbitrary constants. Substitution of (2.16) into either (2.10) or (2.13) demands that  $k = 0$  for consistency. We hence obtain the solution

$$\Omega(y) = \frac{1}{3}\Lambda(y - y_*)^2, \quad (2.17)$$

which is unique up to a fiducial value of  $x^4 = y$ , namely  $y_*$ . This means that the 4D conformal prefactor in (2.2) is fixed by the field equations (2.8). The absorbable constant  $y_*$  notwithstanding, (2.17) defines what are called canonical coordinates in the literature [10,12–14]. Also, note that we need to restrict  $\Omega(y) > 0$  to ensure that the 5D metric (2.2) is well-behaved, which means that  $\Lambda > 0$ . Hence, the 4D sub-manifold represents de Sitter, not anti-de Sitter, spacetimes.

So, we have shown that in the case where the 5D manifold is Ricci-flat there is a unique solution for the conformal prefactor  $\Phi(y)$ , which corresponds to the usual 5D canonical metric (2.1). This solution induces a stress-energy tensor on  $y = \text{constant}$  hypersurfaces consistent with 4D general relativity in the presence of a non-zero cosmological constant and in the absence of ordinary matter.

### III. KILLING VECTORS

In this section, we will derive the form of the Killing vectors of the 5D warped-product space described by the line element (2.2). We write 5D Killing vectors as

$$\xi^A = (\Omega^{-1}\xi^\alpha, -\xi_4) \quad (3.1)$$

$$\xi_A = (\xi_\alpha, \xi_4), \quad (3.2)$$

where  $\xi_\alpha = g_{\alpha\beta}\xi^\beta$ . We will need the Christoffel symbols of the 5D manifold, which we denote by  $\hat{\Gamma}_{BC}^A$ . They are:

$$\begin{aligned} \hat{\Gamma}_{\beta\gamma}^\alpha &= \Gamma_{\beta\gamma}^\alpha, \\ \hat{\Gamma}_{4\beta}^\alpha &= \frac{1}{2}\Omega^{-1}\Omega'\delta_\beta^\alpha, \\ \hat{\Gamma}_{\alpha\beta}^4 &= \frac{1}{2}\Omega'g_{\alpha\beta}, \\ \hat{\Gamma}_{44}^\alpha &= \hat{\Gamma}_{4\alpha}^4 = \hat{\Gamma}_{44}^4 = 0 \end{aligned} \quad (3.3)$$

with

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\sigma}(g_{\beta\sigma,\gamma} + g_{\gamma\sigma,\beta} - g_{\beta\gamma,\sigma}), \quad (3.4)$$

where a comma denotes partial differentiation.

The 5D Killing equation is

$$0 = \hat{\nabla}_A \xi_B + \hat{\nabla}_B \xi_A, \quad (3.5)$$

where  $\hat{\nabla}_A$  is the 5D covariant differential operator. This equation can be split up into three sets of equations in a manner analogous to the splitting of  $R_{AB} = 0$ :

$$0 = \hat{\nabla}_\alpha \xi_\beta + \hat{\nabla}_\beta \xi_\alpha, \quad (3.6)$$

$$0 = \hat{\nabla}_\alpha \xi_4 + \hat{\nabla}_4 \xi_\alpha, \quad (3.7)$$

$$0 = \hat{\nabla}_4 \xi_4. \quad (3.8)$$

From the third equation (3.8), we find

$$\partial_4 \xi_4 = 0 \quad \Rightarrow \quad \xi_4 = \Psi(x^\mu), \quad (3.9)$$

where  $\partial_4 = \partial/\partial y$  and  $\Psi(x^\mu)$  is a 4D scalar function independent of  $y$ . Using this fact and the Christoffel symbols (3.3), equation (3.7) becomes

$$\partial_\alpha \Psi = -\Omega \partial_4 (\Omega^{-1} \xi_\alpha), \quad (3.10)$$

where  $\partial_\alpha = \partial/\partial x^\alpha$ . We can apply the 4D covariant derivative  $\nabla_\beta$  to this result and note that  $\nabla_\beta \partial_\alpha \Psi = \nabla_\alpha \partial_\beta \Psi$  to get

$$0 = \partial_4 [\Omega^{-1} (\nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha)]. \quad (3.11)$$

Now, we can expand and rewrite equation (3.6) to give

$$\mathcal{L}_\xi g_{\alpha\beta} = \Omega' g_{\alpha\beta} \Psi, \quad (3.12)$$

where

$$\mathcal{L}_\xi g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha. \quad (3.13)$$

We will now assume that  $\Psi \neq 0$  and show that this leads to a contradiction unless  $\Omega(y)$  has a specific form. If we take equation (3.12), divide by  $\Omega$ , differentiate with respect to  $y$  and then contract over the spacetime indices, we obtain

$$\Psi^{-1} \square \Psi = -2\Omega \partial_4 (\Omega^{-1} \Omega'), \quad (3.14)$$

where  $\square \equiv \nabla^\alpha \nabla_\alpha$  and we have made use of (3.10). The left-hand side of (3.14) is a function of  $x^\alpha$  only, while the right-hand side is a function of  $y$  only. By separation of variables, we obtain

$$0 = (\square + 2k_1) \Psi, \quad (3.15)$$

$$\frac{k_1}{\Omega} = \frac{d^2}{dy^2} \ln \Omega, \quad (3.16)$$

where  $k_1$  is a constant. The second of these formulae represents a second-order ODE that must be satisfied by the conformal prefactor in order to find a solution to Killing's equation with  $\Psi \neq 0$ . We can integrate (3.16) once to obtain

$$\frac{d}{dy} \ln \Omega = k_1 f(y) + k_2, \quad (3.17)$$

where  $k_2$  is a constant and

$$f(y) = \int^y \Omega^{-1}(u) du. \quad (3.18)$$

We can also integrate (3.10) with respect to  $y$  by introducing an arbitrary dual vector field  $\eta_\alpha(x^\mu)$  that is independent of the fifth coordinate. This gives

$$\xi_\alpha(x^\mu, y) = \Omega(y)\eta_\alpha(x^\mu) - f(y)\Omega(y)\partial_\alpha\Psi(x^\mu). \quad (3.19)$$

Putting this into (3.12) yields

$$\mathcal{L}_\eta g_{\alpha\beta} = 2f(y)\nabla_\alpha\partial_\beta\Psi + g_{\alpha\beta}\Psi\partial_4\ln\Omega. \quad (3.20)$$

Contracting and making use of (3.15) gives

$$\Psi^{-1}\nabla^\alpha\eta_\alpha = 2(\partial_4\ln\Omega - k_1). \quad (3.21)$$

By separation of variables, we require

$$k_3 = \frac{d}{dy} \ln\Omega, \quad (3.22)$$

where  $k_3$  is a constant. Solving this equation gives

$$\Omega(y) = \Omega_0 \exp(k_3 y). \quad (3.23)$$

So, unless the conformal prefactor is given by the above equation, it is impossible to solve Killing's equation with  $\Psi \neq 0$ . Therefore, we must set  $\Psi = 0$  for  $\Omega(y) \neq \Omega_0 \exp(k_3 y)$ .

Setting  $\Psi = 0$  in (3.10) and integrating with respect to  $y$  yields

$$\xi_\alpha = \Omega(y)\eta_\alpha(x^\mu). \quad (3.24)$$

Putting this into (3.12) with  $\Psi = 0$  gives

$$\nabla_\alpha\eta_\beta + \nabla_\beta\eta_\alpha = 0. \quad (3.25)$$

Hence, Killing vectors of our 5D manifold are given by

$$\xi_A = (\Omega(y)\eta_\alpha(x^\mu), 0), \quad \mathcal{L}_\eta g_{\alpha\beta} = 0, \quad (3.26)$$

provided that  $\Omega(y) \neq \Omega_0 \exp(k_3 y)$ . We remark that the 5D Killing vectors are simply related to the 4D Killing vectors  $\eta_\alpha$  of the conformal metric  $g_{\alpha\beta}$ . This fact will be examined more closely below in section V B.

#### IV. THE TRAJECTORY OF 5D NULL PARTICLES

The affinely-parameterized geodesics of the above manifold (2.2) can be derived from the variation of the Lagrangian

$$\mathcal{L} = \frac{1}{2} (d\mathcal{S}/d\lambda)^2 = \frac{1}{2} [\Phi^2 g_{\alpha\beta} k^\alpha k^\beta - \dot{y}^2] \quad (4.1)$$

provided we choose  $k^A k_A = \text{constant}$ . [If we work in a parameterization where the norm of the 5-velocity is variable, we need to extremize  $\int d\mathcal{S}$  instead of  $\int (d\mathcal{S}/d\lambda)^2 d\lambda$ .] Here,  $k^A \equiv dx^A/d\lambda$ ,  $k^\alpha \equiv dx^\alpha/d\lambda$ ,  $\dot{y} \equiv dy/d\lambda$  and  $\lambda$  is an *affine* parameter. The momenta are

$$\begin{aligned} p_\alpha &= \partial\mathcal{L}/\partial k^\alpha = \Phi^2 g_{\alpha\beta} k^\beta, \\ p_4 &= \partial\mathcal{L}/\partial \dot{y} = -\dot{y}. \end{aligned} \quad (4.2)$$

To get the equations of motion, we can use the Euler-Lagrange equations:

$$\frac{d}{d\lambda} \left( \frac{\partial\mathcal{L}}{\partial k^A} \right) - \frac{\partial\mathcal{L}}{\partial x^A} = 0. \quad (4.3)$$

After some algebra, the 4D part of these can be written as

$$^{(\lambda)}a^\alpha = -2\Phi^{-1}\Phi' \dot{y} k^\alpha. \quad (4.4)$$

Here and henceforth, we use the notation  $^{(z)}a^\alpha$  to denote the 4D acceleration in the  $z$  parameterization:

$$^{(z)}a^\alpha \equiv \frac{d^2 x^\alpha}{dz^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{dz} \frac{dx^\gamma}{dz}. \quad (4.5)$$

In equation (4.4),  $z = \lambda$ . Equation (4.4) shows that for an affine parameter  $\lambda$  along the path, there is a velocity-dependent fifth force. The fifth part of (4.3) gives

$$\ddot{y} = -\Phi\Phi' k_\gamma k^\gamma, \quad (4.6)$$

which shows that, in general, the particle accelerates in the fifth dimension.

To continue, we need to choose the type of 5D geodesic we are dealing with. In 4D relativity, the relation  $p^\alpha p_\alpha = m^2$  implies that  $u^\alpha u_\alpha = 1$  for massive particles. In 5D, a natural extension of the 4D energy-momentum relation is  $p^A p_A = 0$  with the fifth component of the momentum being interpreted as the particle mass  $p_4 \sim m$ . This implies that 5D trajectories are null, which is the hypothesis that we will work with in the rest of this paper. Therefore, let us put  $d\mathcal{S}^2 = 0$  or  $k_A k^A = 0$  for null paths. With the Lagrangian chosen as (4.1), the equations (4.4) and (4.6) are still well defined. In this case, the metric (2.2) gives

$$\dot{y}^2 = \Phi^2 k_\gamma k^\gamma. \quad (4.7)$$

We can use (4.7) to substitute for  $k_\gamma k^\gamma$  ( $\neq 1$ ) in (4.6). This gives  $\ddot{y}/\dot{y}^2 = -\Phi^{-1}d\Phi/dy$ , which is solved by

$$\dot{y} = K\Phi^{-1}(y). \quad (4.8)$$



Here  $K$  is a constant. We can integrate this result noting that  $Kd\lambda = \Phi(y)dy$ , so in terms of two other constants  $\lambda_0$  and  $y_0 = y(\lambda_0)$ , we have

$$K(\lambda - \lambda_0) = \int_{y_0}^y \Phi(u)du. \quad (4.9)$$

We now put (4.8), which depends on the null assumption and the equation of motion in the fifth dimension, into the equations of motion in 4D (4.4), to obtain

$$^{(\lambda)}a^\alpha = -2K\Phi^{-2}\Phi'k^\alpha. \quad (4.10)$$

Also, (4.8) back into (4.7) gives

$$K^2 = \Phi^4 k_\gamma k^\gamma. \quad (4.11)$$

The relations (4.10) and (4.11) describe paths in  $(4+1)$ D in terms of a parameter (4.9) which is an integral over the conformal factor associated with the 4D part of the metric.

Our geodesics depend on three arbitrary parameters  $\lambda_0$ ,  $K$  and  $y_0$ . We can remove the former two from the analysis by performing a transformation of the affine parameter:  $\lambda \rightarrow \tilde{\lambda} = \lambda/|K| + \lambda_0$ , provided  $K \neq 0$ . Since  $d\tilde{\lambda}/d\lambda > 0$ , this transformation preserves the orientation of the 5D null curve. We can include the  $K = 0$  case explicitly by defining  $\epsilon \equiv K/|K| = \pm 1$  when  $K \neq 0$  and  $\epsilon = 0$  when  $K = 0$ . Dropping the tilde on the new parameter, we find that (4.9), (4.10) and (4.11) become

$$\epsilon\lambda = \int_{y_0}^y \Phi(u)du, \quad (4.12)$$

$$^{(\lambda)}a^\alpha = -2\epsilon\Phi^{-2}\Phi'k^\alpha, \quad (4.13)$$

$$\epsilon^2 = \Phi^4 k_\gamma k^\gamma. \quad (4.14)$$

These are the relations we will be concerned with in what follows. Notice that if  $\epsilon = 0$ , equation (4.12) implies that  $y = y_0$  for all  $\lambda$ , which means that there is no motion in the fifth dimension.

To see where (4.12)–(4.14) lead, let us perform a parameter transformation  $\tau = \tau(\lambda)$ . Then  $k^\alpha = v^\alpha d\tau/d\lambda$ , where  $v^\alpha \equiv dx^\alpha/d\tau$ . Equations (4.13) and (4.8) give

$$^{(\tau)}a^\alpha = -\left(\frac{d\lambda}{d\tau}\right)^2 \left(\frac{d^2\tau}{d\lambda^2} + \frac{2}{\Phi} \frac{d\Phi}{dy} \frac{dy}{d\lambda} \frac{d\tau}{d\lambda}\right) v^\alpha. \quad (4.15)$$

Clearly, we can choose  $\tau = \tau(\lambda)$  in such a way as to make the right-hand side of (4.15) zero. This happens if  $d\tau/d\lambda = C/\Phi^2$ , where  $C$  is a dimensionless constant we can set equal to unity. The 4D motion is described by

$$^{(\tau)}a^\alpha = 0, \quad (4.16)$$

which is the standard geodesic equation, provided that

$$d\tau/d\lambda = \Phi^{-2}. \quad (4.17)$$

This into (4.14) gives

$$\epsilon^2 = v^\alpha v_\alpha. \quad (4.18)$$

Hence, we see that  $v^\alpha(\tau)$  is a 4D geodesic of  $g_{\alpha\beta}$  that can be timelike ( $\epsilon = \pm 1$ ) or null ( $\epsilon = 0$ ). Further, this implies that the parameter  $\tau$  is either the proper time or an affine null parameter along the 4D path in the conformal spacetime described by  $g_{\alpha\beta}$ . It is for this reason that we call  $\tau$  the *conformal proper time*. Also, (4.8) with (4.17) shows that  $dy/d\tau = \epsilon\Phi(y)$ , so

$$\epsilon\tau = \int_{y_0}^y \Phi^{-1}(u) du. \quad (4.19)$$

We see that this relation (4.19) replaces (4.12), the geodesic (4.16) replaces (4.13), and (4.18) replaces (4.13). In other words, particles which move on null 5D paths have trajectories that are in accordance with the conventional 4D geodesic equation, if the parameter is judiciously chosen. This brings the physical relevance of the extra force term in equation (4.13) into question. It is important to note from (4.18) that even though the path is 5D null, it is not necessarily 4D null. Massive ( $v_\alpha v^\alpha > 0$ ) or massless ( $v_\alpha v^\alpha = 0$ ) particles in 4D can move on null paths in 5D.

This remarkable result holds irrespective of the form of  $\Phi = \Phi(y)$  in (2.2). However, to make contact with previous work [10,12–14], let us choose the canonical form  $\Phi(y) = y/L$ . Then,  $dy/d\tau = \epsilon y/L$  and  $y = y_0 e^{\epsilon\tau/L}$ , where  $y_0$  is a constant. By (2.2),  $d\mathcal{S}^2 = 0$  but  $\sqrt{v_\alpha v^\alpha} \neq 0$ .

There is yet another choice of parameter that we ought to consider. This third parameter choice is based on the induced metric and is defined by

$$\epsilon^2 = h_{\alpha\beta} u^\alpha u^\beta = \Phi^2 g_{\alpha\beta} u^\alpha u^\beta, \quad (4.20)$$

where  $u^\alpha = dx^\alpha/ds$ . This parameterization enforces the proper normalization of the particle trajectory for observers confined to  $\Sigma_y$  hypersurfaces. It is for this reason that we call  $s$  the *hypersurface proper time*. The  $s$ -parameterization is preferred by Liu and Mashhoon [15]. Examining equation (4.14), we see that we can satisfy the hypersurface-normalization condition (4.20) by setting

$$ds/d\lambda = \Phi^{-1}. \quad (4.21)$$

Under such a transformation, we can use a formula analogous to (4.15) with (4.8) to derive the 4D part of the geodesic equation:

$${}^{(s)}a^\alpha = -\epsilon\Phi^{-1}\Phi' u^\alpha. \quad (4.22)$$

We see that in this parameterization we have a velocity-dependent extra force acting on the particle. In canonical coordinates where  $\Phi(y) = y/L$ , the right-hand side of (4.22) becomes  $\epsilon u^\alpha/(s + \epsilon)$ , i.e. it decreases with increasing proper time. This result represents a deviation from geodesic motion as measured by observers on  $\Sigma_y$ . Transforming our solution for  $\dot{y}$  (4.8) gives

$$dy/ds = \epsilon \quad \Rightarrow \quad y(s) = \epsilon s + y_0. \quad (4.23)$$

That is, the particle has a constant velocity in the  $y$  direction which we have normalized to  $\pm 1$  or  $0$ . (This is in agreement with the  $K = 0$  case presented by Liu & Mashhoon [15].) We again discover that if the 4D path is null ( $\epsilon = 0$ ), the particle is confined to  $\Sigma_y$ .

The three types of parameterization that we have discussed in this section are summarized in Table I. Of the three scenarios, the conformal parameterization most resembles what we are used to in 4D physics. This might tempt us to decide that the conformal parameterization is the “correct” choice. However, such an identification would be premature. The preferred parameter in 4D general relativity is the proper time, which has the geometric interpretation of being the arclength along timelike geodesics and the physical interpretation of being the time measured by freely-falling clocks. In our 5D picture there exists no useful notion of 5D arclength because the particle trajectories are null — we only have the 5D affine parameter  $\lambda$ . We have encountered two equally valid notions of 4D arclength: the proper time in the 4D conformal manifold ( $\tau$ ) and the proper time associated with the projection of geodesics onto a  $\Sigma_y$  hypersurface ( $s$ ). The only way to distinguish between these choices is to study the physics associated with each, which is what we do in the following section.

## V. THE PHYSICAL PROPERTIES OF THE TRAJECTORIES

When particles follow higher-dimensional geodesic paths, they often seem to have peculiar physical properties as measured by 4D observers. For example, it has been observed by many authors that particles following geodesic paths in higher dimensions seem to have variable rest masses according to observers ignorant of the extra dimensions [12,15]. We propose to examine the physical properties of the trajectories derived in the previous section and hence determine what characteristics of the dynamics are observationally testable.

### A. Rest mass variations and a variable cosmological “constant”

We want to analyze how an observer ignorant of the fifth dimension might interpret kinematic data concerning the trajectory of freely-falling observers in a 5D manifold. When reducing observational data, such observers are likely to fall back on the 4D relativistic version of Newton’s second law. That is, they will demand that the particle’s 4-momentum  $p^\alpha$  must be covariantly conserved in a 4D sense:

$$\frac{Dp^\alpha}{dz} \equiv \frac{dp^\alpha}{dz} + \Gamma_{\beta\gamma}^\alpha U^\beta p^\gamma = 0. \quad (5.1)$$

Here  $z$  stands for whatever parameter we are using along the path ( $\lambda$ ,  $\tau$ ,  $s$ , etc ... ) and the 4-momentum is assumed to have the standard form

$$p^\alpha = mU^\alpha, \quad U^\alpha = \frac{dx^\alpha}{dz}, \quad (5.2)$$

where  $m$  is the mass. Let us expand (5.1), assuming that the mass varies with  $z$ . We obtain

$${}^{(z)}a^\alpha = -\frac{1}{m} \frac{dm}{dz} U^\alpha. \quad (5.3)$$

If we compare this formula with the results presented in Table I, we come to a disturbing conclusion: a particle's mass variation depends explicitly on the choice of parameterization. For example, it is easy to see that if the particle's world line is parameterized by the 5D affine parameter  $\lambda$ , the particle mass is given by

$$m(\lambda) = k\Phi^2(y(\lambda)), \quad (5.4)$$

where  $k$  is a constant. We can either view the particle mass as a function of  $\lambda$  or as a function of  $y$ . If we put (5.4) into the normalization condition (4.14), we obtain

$$\epsilon^2 k^2 = g_{\alpha\beta}(mk^\alpha)(mk^\beta) = g_{\alpha\beta}p^\alpha p^\beta. \quad (5.5)$$

Hence, the norm of the four momentum (as defined by the 4D conformal metric) is conserved along the worldline, which follows from the fact that  $Dp^\alpha/d\lambda = 0$ . This is despite the fact that the norm of the 4-velocity is not constant (the variation in mass precisely cancels that effect). Our initial assumption (5.1) made no particular choice of 4D metric, yet  $g_{\alpha\beta}$  has been singled out by this calculation. Now if we chose to raise and lower indices with the induced metric  $h_{\alpha\beta}$ , the norm of  $p^\alpha$  would be variable, suggesting that the conformal metric defines the line element appropriate to observers unaware of the fifth dimension. For the canonical prefactor  $\Phi(y) = y/L$ , (5.4) gives

$$m(\lambda) = 2kL^{-1}\epsilon^2\lambda, \quad (5.6)$$

where we have chosen  $y(\lambda = 0) = 0$ . We see that in the affine parameterization, the mass increases linearly in “time”. However, the variation is small if  $L$  is large, or the induced cosmological constant  $\Lambda$  is small. If  $\epsilon = 0$ , we recover that massless particles travel on 4D null geodesics.

Does this interpretation hold up in the hypersurface parameterization? The mass function in this case is given by

$$m(s) = k\Phi(\epsilon s + y_0). \quad (5.7)$$

Again, the mass may be viewed as a function of  $y = y(s)$ . The normalization condition (4.20) yields, as before:

$$\epsilon^2 k^2 = g_{\alpha\beta}(mk^\alpha)(mk^\beta) = g_{\alpha\beta}p^\alpha p^\beta. \quad (5.8)$$

For the canonical prefactor, we obtain

$$m(s) = kL^{-1}(\epsilon s + y_0). \quad (5.9)$$

The mass is constant if  $\epsilon = 0$ , and is zero if  $k = 0$  also. We note that mass variations are small if the induced cosmological constant is small.

Finally, we can deal with the trivial case of conformal parameterizations. Since the 4D equation of motion (4.16) is precisely affinely geodesic, there is no mass variation in this parameterization. This follows from that fact that the 4-velocity is normalized to have a constant length, which means that the mass must also be constant to ensure that  $p^\alpha p_\alpha$  is conserved.

We see that there are three different masses for the three different parameterizations. However, the conformal metric has been singled out as the 4D metric appropriate to observers ignorant of  $y$  (as opposed to the induced metric, which is appropriate to observers confined to  $\Sigma_y$  hypersurfaces). In this parameterization, particle masses are constant.

It therefore becomes obvious that for an arbitrary parameterization  $U^\alpha = dx^\alpha/dz$  related to the affine parameterization by a transformation of the form  $dz/d\lambda = G(y(\lambda))$ , where  $G$  is some function of  $y$ , the mass is defined by the normalization relation

$$\epsilon^2 k^2 = m^2(y) g_{\alpha\beta} U^\alpha U^\beta, \quad (5.10)$$

where  $k$  is a constant. This has an interesting interpretation when the 5D vacuum field equations  $R_{AB} = 0$  are enforced. From equation (2.15), the induced 4D stress-energy tensor is

$$8\pi T_{\alpha\beta} = \Lambda g_{\alpha\beta}. \quad (5.11)$$

Now, the energy density  $\rho$  of cosmological matter will be measured by an observer with 4-velocity  $U^\alpha$  to be

$$8\pi\rho = 8\pi T_{\alpha\beta} U^\alpha U^\beta = \epsilon^2 \Lambda k^2 m^{-2}(y). \quad (5.12)$$

Hence, there is a direct relation between the energy density of the vacuum and the mass of the particle. If the particle mass varies, an observer traveling along with the particle will measure the energy density of the vacuum to be variable. That is, the observer will measure a variable cosmological “constant”. We can consider small changes in the particle mass  $\delta m$  connected with small changes in the energy density  $\delta\rho$ :

$$\delta m = -4\pi\Lambda^{-1}k^{-2}m^3\delta\rho, \quad (5.13)$$

where we have taken  $\epsilon^2 = 1$ . This has the suggestive form of an energy conservation equation. Let us assume that the particle has a 3D “volume” associated with it that is related to its mass  $V = V(m)$  [as in the black hole case]. Let us also assume that a change in vacuum energy  $\delta E$  in the volume occupied by the particle results in an increase or decrease of the particle mass:  $\delta m = -\delta E$ . However, we have  $\delta E = \delta(\rho V)$ . Using these relations we can derive a differential equation for  $dV/dm$ :

$$0 = m dV/dm - 2V - 8\pi\Lambda^{-1}k^{-2}m^3, \quad (5.14)$$

where we have cancelled a common factor of  $\delta m$ . Setting  $V(m=0) = 0$ , which implies that massless particles remain massless, we get

$$V(m) = 8\pi\Lambda^{-1}k^{-2}m^3. \quad (5.15)$$

Therefore, as particles move through the 5D manifold they will in general observe the cosmological “constant” to be varying in time. Further, if one assumes that the particle occupies a 3D volume of linear dimension  $\sim m$  (as is the case for a black hole) then the energy being gained or lost by the vacuum corresponds to the decrease or increase of the particle’s mass.

## B. Constants of the motion and the particle energy

The fact that the definition of rest mass is parameter-dependent may be considered by some to be unsatisfactory. A physical quantity like  $m$  should be independent of the timing mechanism employed to separate points along the particle's worldline. To remove the ambiguity in parameterization, we attempt to construct observable quantities that depend only on the 5D coordinates and not the parameter. A physically meaningful class of observables for spacetimes with a certain degree of symmetry are the constants of the motion, such as the energy, linear momentum, angular momentum, etc . . . . We can argue that such quantities ought to be independent of  $y$ , which implies that there is no intrinsic rest-mass variation.

Let us assume that the conformal manifold admits the existence of a Killing vector  $\eta^\alpha$  such that  $\mathcal{L}_\eta g_{\alpha\beta} = 0$ . Then by the results of section III, the 5D manifold has a Killing vector of the form

$$\xi^A = (\eta^\alpha, 0). \quad (5.16)$$

We exclude the special case  $\Omega(y) = \Omega_0 \exp(k_3 y)$ , so all 5D Killing vectors are of the form (5.16). Now, since  $k^B \nabla_B k^A = 0$ , we have that

$$\mathcal{K}_\eta = \xi_A k^A \quad (5.17)$$

is a constant of the motion. Here  $k^A = dx^A/d\lambda$ . We would like to write  $\mathcal{K}_\eta$  in a form independent of the parameter. To do so, we introduce a time foliation of the 4D part of the manifold. This allows us to write the conformal line element in lapse and shift form:

$$\begin{aligned} d\tau^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{00} [dt^2 - \sigma_{ij} (N^i dt + dx^i)(N^j dt + dx^j)], \end{aligned} \quad (5.18)$$

where  $i, j = 1, 2, 3$ . Here,  $g_{00}$  is the redshift factor,  $N^i$  is the shift 3-vector and  $g_{00}\sigma_{ij}$  is the 3-metric. We can use the normalization condition for the affine parameterization (4.14) (with  $\epsilon = 1$ ) to obtain

$$\frac{d\lambda}{dt} = \Phi^2(y) g_{00}^{1/2} (1 - \beta^2)^{1/2}, \quad (5.19)$$

where

$$\beta^2 = \sigma_{ij} (N^i + V^i)(N^j + V^j), \quad (5.20)$$

with  $V^i = dx^i/dt$ . By an appropriate choice of foliation we can set  $N^i = 0$ , which reduces  $(1 - \beta^2)^{1/2}$  to the Lorentz factor  $(1 - V^2)^{1/2}$  when  $g_{00} = 1$ . Therefore, we may write  $\mathcal{K}_\eta$  as

$$\mathcal{K}_\eta = \frac{g_{\alpha\beta} \eta^\alpha V^\beta}{g_{00}^{1/2} (1 - \beta^2)^{1/2}}, \quad V^\alpha = \frac{dx^\alpha}{dt}. \quad (5.21)$$

This form is independent of the parameter choice used to solve the 5D geodesic equation. It is also independent of the extra dimension  $y$  by virtue of the fact that  $g_{\alpha\beta}$  and  $\eta^\alpha$  are functions of spacetime variables only. Therefore, if observers can measure the 4-dimensional

position of a freely-falling particle at different points along its world line, they can construct the constants of the motion without knowledge of  $dy/d\lambda$  or  $dy/dt$ .

We can illustrate this point by considering a specific 4D metric which corresponds to a solution of the 5D vacuum field equations, namely the Schwarzschild–de Sitter one:

$$g_{\alpha\beta}dx^\alpha dx^\beta = f(r) dt^2 - f^{-1}(r) dr^2 - r^2 d\Omega^2, \\ f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2. \quad (5.22)$$

Here,  $M$  is the mass while  $\Lambda$  is the induced cosmological constant (*cf.* section II). This spacetime is static and spherically symmetric, so it has a timelike Killing vector  $\tau^\alpha = (\partial/\partial t)^\alpha$  that we can use to define the energy, and an azimuthal Killing vector  $\phi^\alpha = (\partial/\partial \phi)^\alpha$  we can use to define the angular momentum. For an equatorial orbit ( $\theta = \pi/2$ ), the angular momentum (up to a multiplicative constant) is

$$L = \frac{f^{-1/2}(r)r^2}{(1 - \beta^2)^{1/2}} \frac{d\phi}{dt}, \quad (5.23)$$

where  $\beta^2 = r^2 f^{-1}(r) d\phi/dt$ . If the mass of the particle varies with  $y$ , we would expect an additional function of  $y$  to appear in this expression. That is, if  $m$  changes as the particle moves between  $\Sigma_y$  hypersurfaces, then we physically expect either the orbital velocity  $d\phi/dt$  or the particle's radial position  $r$  to change in a fashion that leaves  $L$  constant. However, equation (5.21) implies that the particle's 4D worldline is insensitive to motion in the  $y$  direction, which is a direct consequence of the conformal equation of motion (4.16). This tends to support the view that the particle mass is constant. This argument is not restricted to spacetimes with azimuthal symmetry, since the general form of the constants of motion (5.21) is general.

Using  $\tau^\alpha = (\partial/\partial t)^\alpha$ , we can also define the particle's energy. Let us take  $1 \gg 2M/r \gg \Lambda r^2/3$  and consider only radial motion  $d\theta/dt = d\phi/dt = 0$ . Then to first order in  $M$  and  $v_r = dr/dt$ , and zeroth order in  $\Lambda$ , the energy is

$$E = 1 + \frac{v_r^2}{2} - \frac{M}{r} + \dots \quad (5.24)$$

The second and third terms are obviously the Newtonian kinetic and potential energies, which means that the first term must be the rest mass energy. The fact that the rest mass energy is a constant independent of  $y$  confirms that the rest mass does not vary along the particle's world line, at least according to an analysis based on constants of the motion.

### C. 5D time dilation

While the calculation of the previous subsection has the advantage of being independent of the parameterization of the trajectory, it has the disadvantage of being a coordinate-dependent manipulation that relies heavily on our choice of foliation. That is, the  $dt$  coordinate time interval is not an invariant quantity. However, the 4D proper time interval is indeed an invariant under 4D coordinate transformations, which suggests that we ought to

write our equations in terms of  $dz$ . The problem is that each of the parameterizations  $\lambda$ ,  $\tau$  or  $s$  (and others) could qualify as the proper time  $z$ . To our knowledge, there is no *a priori* method for determining the “true” 4D proper time; but it is easy to imagine an experiment which would show which one is most convenient.

Consider a spherically-symmetric spacetime that allows for circular orbits [like (5.22) above]. By virtue of equations (4.16) and (4.18), the conformal time interval  $\Delta\tau$  associated a complete revolution in a circular orbit is constant (i.e. the orbital velocity  $d\phi/d\tau$  is constant). Now, suppose that we have a satellite in a circular orbit that carries an atomic clock or some other time-keeping device. This clock measures the proper time along the path by “ticking”  $\Delta N/\gamma$  times during a proper time interval  $\Delta z$  ( $\gamma$  is the constant rate at which the clock oscillates). As seen above, the relation between different 4D parameterizations is in general given by

$$d\tau/dz = F(y(\tau)), \quad (5.25)$$

where  $F$  is some function of  $y$ , so that

$$\Delta N = \gamma \int_{\tau_i}^{\tau_i + \Delta\tau} F^{-1}(y(\tau)) d\tau. \quad (5.26)$$

Here  $\tau_i$  is when we start keeping time and also represents the initial  $y$  position of the circular orbit. Now, let us count the number of oscillations  $\Delta N_1$  that our clock undergoes during a complete orbit starting at time  $\tau_1$ , and then repeat the procedure for another orbit starting at a later time  $\tau_2$ . If we adopt the canonical prefactor  $\Phi(y) = y/L$ , the ratio of the number of clock oscillations during the two orbits is

$$\frac{\Delta N_2}{\Delta N_1} = \begin{cases} e^{2\epsilon(\tau_2 - \tau_1)/L}, & z = \lambda \\ 1, & z = \tau \\ e^{\epsilon(\tau_2 - \tau_1)/L}, & z = s. \end{cases} \quad (5.27)$$

Here the time parameter  $\tau$  is related to  $y$  via

$$y(\tau) = y_0 e^{\epsilon\tau/L}. \quad (5.28)$$

Therefore, if the 4D proper time is not  $\tau$  then the time elapsed in the rest frame of the orbiting body during one complete revolution will not be constant. That is, an observer moving with the clock will conclude that the clock is speeding up or slowing down (depending on whether they are moving in the direction of increasing or decreasing  $y$ ); or that the orbital velocity  $d\phi/dz \approx 2\pi\gamma/\Delta N$  is growing smaller or larger with time. Of course, the effect is small if  $L$  is large or  $\Lambda$  is small. These effects are in principle testable, and could be used to distinguish between possible candidates for the best proper time.

## VI. CONCLUSION

To better understand the dynamics of particles moving in a higher-dimensional world but observed in spacetime, we have introduced a 5D warped-product space which is related to the 4D sub-manifold via a conformal factor that depends on the extra coordinate.



When the 5D field equations are the standard (vacuum) ones of Kaluza-Klein theory, the 4D sub-manifold represents spacetimes with a non-zero cosmological constant and devoid of ordinary matter. We have examined the 5D Killing vectors, which are related to the 4D Killing vectors of the sub-manifold. A major result is that null geodesics in 5D can correspond to non-null geodesics in 4D. That is, massless particles in Kaluza-Klein space can correspond to massive particles in Einstein space. It has been shown that there is in general an anomalous acceleration in the 4D equation of motion that can be removed by a parameter transformation. This brings up the question of how the “true” proper time, which is the time measured by freely falling clocks, should be chosen. The ambiguity in the choice of parameterization results in multiple expressions for the particle mass, which in general vary along the particle’s worldline. In the 5D vacuum case, the variation in rest mass can be related to the variation in the vacuum energy as measured by an observer traveling with the particle. However, we have shown that the constants of the motion can be written in a form independent of both the parameter and the extra coordinate, which suggests to us that variable rest mass may be an artifact of a poor choice of parameter. We have argued that the best choice of the parameter that describes a particle’s motion, and the question of the variability of its rest mass, can be tested by experiment or observation.

In closing, we should remind ourselves that modern Kaluza-Klein theory (without the cylinder and compactification condition) is fully covariant in 5D. One can argue that the same requirements be made of superstrings in 10D and supergravity in 11D, and this is indeed a strength of much recent work on brane theory in  $ND$ . However, we currently interpret experimental and observational data in terms of four spacetime dimensions. Therefore, to make contact with everyday experience we naturally attempt to interpret 5D geometric objects, like null geodesics, within the context of 4D spacetime. This dimensional reduction is the reason that massless particles in 5D can appear to have (possibly variable) finite rest masses in 4D. However, the details of the reduction from 5D to 4D are not unique, so we suggest that further work be done to determine the most convenient reduction scheme.

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# TABLES

TABLE I. Summary of the different type of parameterization of 5D null geodesics

Parameterization	4D Normalization	4D Equation of Motion	Motion in $y$ direction
Affine ( $\lambda$ )	$\epsilon^2 = \Phi^4 g_{\alpha\beta} k^\alpha k^\beta$	$^{(\lambda)}a^\alpha = -2\epsilon\Phi^{-2}\Phi'k^\alpha$	$\epsilon\lambda = \int^y \Phi(u) du$
Conformal ( $\tau$ )	$\epsilon^2 = g_{\alpha\beta} v^\alpha v^\beta$	$^{(\tau)}a^\alpha = 0$	$\epsilon\tau = \int^y \Phi^{-1}(u) du$
Hypersurface ( $s$ )	$\epsilon^2 = h_{\alpha\beta} u^\alpha u^\beta$	$^{(s)}a^\alpha = -\epsilon\Phi^{-1}\Phi'u^\alpha$	$y(s) = \epsilon s + y_0$